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STATE-SPACE MODELING OF TWO-DIMENSIONAL VECTOR-EXPONENTIAL TRAJECTORIES*

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Abstract. We solve two problems in modeling polynomial vector-exponential trajectories dependent on two independent variables. In the first one we assume that the data-generating system has no inputs, and we compute a state representation of the most powerful unfalsified model for this data. In the second instance we assume that the data-generating system is controllable and quarter-plane causal, and we compute a Roesser input-state-output model. We provide procedures for solving these identification problems, both based on the factorization of constant matrices directly constructed from the data, from which state trajectories can be computed.

Key words. multidimensional systems, most powerful unfalsified model, Roesser models, bilinear differential forms

AMS subject classifications. 93A30, 93B15, 93B20, 93B30, 93C20

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1. Introduction.

We consider two problems in modeling two-dimensional (2D) continuous trajectories from data. In both cases the data consists of polynomial vector-exponential trajectories, and we seek *state-space models* explaining it, i.e., systems of partial differential equations of first order in an auxiliary, "state" variable, and zeroth-order in the measured, "external" variable. The two situations differ in the model class we assume the data-generating system belongs to: in the first case we seek an *autonomous* state model, i.e., a system without inputs; in the second one we assume that an input/output partition of the external variable is given, and we compute an *input-state-output* (i/s/o) model.

Modeling 2D polynomial vector-exponential trajectories with autonomous systems has been considered in [32, 33], on whose results the first part of this paper on the computation of autonomous models heavily relies. Modeling vector-exponential trajectories with transfer-function (i.e., input-output) models is closely related to two-variable rational interpolation; the latter has been investigated in the SISO case in [2]. The approach taken in the present paper differs fundamentally from those: we use data to first compute state trajectories corresponding to it, and in a second stage we compute a state representation for the data and the identified state trajectories by solving linear equations in the unknown state, input, and output matrices.

Modeling methodologies where state-trajectories are computed from data and state-equations are subsequently computed are well-known in the one-dimensional (1D) case as *subspace identification* methods (see, e.g., [12]). Such ideas have been pursued much less frequently in the 2D case: see [7, 19] for a pioneering subspace-identification approach to the computation of i/s/o representations of denominator-separable 2D discrete-systems from data. Our modeling approach differs essentially

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from those, in that it does not exploit the shift-invariance properties of data trajectories, but rather the fact that *external properties*, i.e., properties at the level of external variables, in our case *duality*, are reflected into *internal properties*, i.e., at the level of state. To make the connection between internal and external properties conceptually and computationally accessible we use constant matrices associated with bilinear forms on the system variables and their derivatives. Our approach is related to the 1D *Loewner framework*; we refer to [3, 20] for an introduction to the Loewner framework close in spirit to the ideas illustrated in this paper.

In section 2 we state two modeling problems, and we discuss some of their features and relations with previous work. In section 3 we gather the necessary background material. We illustrate our approach to the computation of state representations for the autonomous case in section 4, where we also state an algorithm to compute a minimal state representation for such models. In section 5 we discuss our framework for the solution of the i/s/o modeling problem. Section 6 contains our concluding remarks, including an overview of issues of current research.

Notation. We denote by $\mathbb{C}^{m\times n}$ the set of all $m \times n$ matrices with entries in $\mathbb{C}$. $\mathbb{C}^{\bullet\times n}$ denotes the set of matrices with n columns and an unspecified (finite) number of rows. Given $A \in \mathbb{C}^{m\times n}$, we denote by A^* its conjugate transpose, and by $A^\dagger$ its Moore–Penrose pseudoinverse. If A, B are matrices with the same number of columns, $\mathrm{col}(A, B)$ is the matrix obtained stacking A on top of B. $\mathbb{C}[\xi_1, \xi_2]$ is the ring of bivariate polynomials in the indeterminates ξ_1, ξ_2 with complex coefficients, and $\mathbb{C}^{m\times n}[\xi_1, \xi_2]$ that of $m \times n$ bivariate polynomial matrices. Similarly, $\mathbb{C}^{m\times n}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ is the ring of $m \times n$ polynomial matrices in the indeterminates ζ_1, ζ_2, η_1, η_2. $\mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$ denotes the space of $\mathbb{C}^{\mathtt{w}}$-valued smooth functions defined on $\mathbb{R}^2$. $e^{\lambda_1 \cdot}e^{\lambda_2 \cdot}$ denotes the function from $\mathbb{R}^2$ to C whose value at (t_1, t_2) is $e^{\lambda_1 t_1}e^{\lambda_2 t_2}$.

2. Problem statement.

We are given a finite set of 2D polynomial vector-exponential trajectories $w_i(\cdot, \cdot)$, whose value at (t_1, t_2) is

$$w_i(t_1, t_2) := \sum_{k_1=0}^{L_1^i} \sum_{k_2=0}^{L_2^i} \overline{w_{k_1,k_2}^i} t_1^{k_1} t_2^{k_2} e^{\lambda_1^i t_1} e^{\lambda_2^i t_2} \ , \ i = 1, \dots, N \ , \tag{2.1}$$

where $\overline{w_{k_1,k_2}^i} \in \mathbb{C}^{\mathtt{w}}$, $k_\ell = 0, \dots, L_\ell^i$, $\ell = 1, 2$ and $\lambda_j^i \in \mathbb{C}$, $i = 1, \dots, N$, $j = 1, 2$, and t_1 and t_2 are two independent continuous variables. In the following we state *two identification problems* that differ from each other in the underlying assumptions on the model class the system generating the data (2.1) belongs to.

2.1. The autonomous case.

It has been shown in [32] that the data (2.1) can be modeled by the *most powerful unfalsified model* (*MPUM*), i.e., the smallest linear subspace $\mathfrak{B}^* \subseteq \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$ closed under differentiation that contains the trajectories (2.1). In [32] it is shown that the MPUM for (2.1) is *autonomous*: there are no *free* components in w, i.e., components which can take arbitrary values in $\mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$; moreover, the MPUM is *finite-dimensional* as a subspace of $\mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$.

In the proof of [32, Thm. 3] an algorithm ultimately based on commutative algebra is provided to construct a *state-representation* of the MPUM, i.e., to compute $\mathtt{n} \in \mathbb{N}$ and matrices $A_i \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$, $i = 1, 2$, and $C \in \mathbb{C}^{\mathtt{w}\times\mathtt{n}}$ such that the MPUM is

$$\mathfrak{B}^* := \Big\{ w \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}}) \mid \exists\, x \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{n}}) \text{ s.t. } \frac{\partial}{\partial t_i} x = A_i x, i = 1, 2 \text{ and } w = Cx \Big\} \ . \tag{2.2}$$

In [32] the matrices A_i, $i = 1, 2$ and C of a (generally nonminimal) representation (2.2) are computed by inspection directly from the data (2.1). In section 4 of this paper we pursue a different approach to compute a *minimal* state representation (2.2) of the MPUM, based on the rank-revealing factorization of a constant matrix obtained from the trajectories w_i and their derivatives at $(t_1, t_2) = (0, 0)$. Such rank-revealing factorization produces the values at $(0, 0)$ of state trajectories $x(\cdot, \cdot)$ associated with the data (2.1) and their partial derivatives. The matrices A_1, A_2, C corresponding to a minimal state-space representation (2.2) of the MPUM can then be computed solving a system of linear equations involving the constructed state trajectories values and the data. An advantage of our approach over the method of [32] is that minimal state representations of the MPUM are obtained *directly* from the data, without any further computation, e.g., the reduction of a precomputed representation to a Kalman observability form suggested in [32, sect. 2].

2.2. The input-output case. The second problem considered in this paper arises when an *input-output partition* $w = \mathrm{col}(u, y)$ of the variables is known, and moreover, the data-generating system is *controllable* (see [31] for a definition) and *quarter-plane causal* (see [27, sect. III]). It is well-known that such a system can be represented by a *Roesser i/s/o representation* (introduced in [28] in the discrete-case):

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial t_1} x_1 \\ \frac{\partial}{\partial t_2} x_2 \end{bmatrix} &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \, , \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du \, , \end{aligned} \tag{2.3}$$

where $A \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$, $B := \mathrm{col}(B_1, B_2) \in \mathbb{C}^{\mathtt{n}\times\mathtt{m}}$, $C := \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{C}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{C}^{\mathtt{p}\times\mathtt{m}}$, and the external variable $w := \mathrm{col}(u, y)$.

In section 5 of this paper we show how to compute matrices A, B, C such that (2.3) are satisfied for some trajectories x_i and the data $w_i = \mathrm{col}(u_i, y_i)$, $i = 1, \ldots, N$ in (2.1). Our approach to this identification problem is based on rank-revealing factorizations of *constant matrices* obtained from the data (2.1) and their *dual trajectories*, i.e., external trajectories of the *dual system*.[1] Such factorizations produce the values at $(0, 0)$ of state trajectories x_i corresponding to the data w_i in some Roesser representation (2.3). Once the $x_i(0, 0)$, $i = 1, \ldots, N$ are known, the matrices A, B, C, and D can be computed in a straightforward way. In our approach an essential role is played by the calculus of *bilinear differential forms* and their representation as four-variable polynomial matrices.

3. Background material. We give only the minimum amount of information needed; see [10, 17, 18] for more information and [11, 13, 14, 21, 23] for important details and for applications of 2D bilinear and quadratic differential forms.

3.1. Two-dimensional systems. A subset $\mathfrak{B}$ of the space $\mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$ of infinitely differentiable trajectories in two independent variables is called a 2D linear differential behavior if it is the solution set of a system of linear, constant-coefficient partial differential equations (in the following PDEs) in two independent variables. That is, $\mathfrak{B}$ is the subset of $\mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$ consisting of all solutions to

$$R\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right) w = 0 \, , \tag{3.1}$$

[1] We show in Remark 5 that dual trajectories do not need to be measured from the dual system but can be computed *directly* from the primal data (2.1) via a technique called *mirroring*.

where R is a polynomial matrix in the indeterminates ξ_i, $i=1,2$. We call (3.1) a *kernel representation* of $\mathfrak{B}$, and we denote the set consisting of all linear differential $2D$-systems with $\mathtt{w}$ external variables with $\mathcal{L}_2^{\mathtt{w}}$.

If $\mathfrak{B}$ is *controllable* (see [18, Def. 1]), then it admits an image representation

$$w = M\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right)\ell\,, \tag{3.2}$$

where $w \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$, the *latent variable* $\ell \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^1)$, and M is a polynomial matrix in the indeterminates ξ_i, $i=1,2$ with a suitable number of columns. Such a set of PDEs represents the *full behavior* $\mathfrak{B}_f \in \mathcal{L}_2^{\mathtt{w}+1}$ defined by

$$\mathfrak{B}_f := \{(w,\ell) \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}+1}) \mid (3.2) \text{ are satisfied}\}$$

and the *external behavior* $\mathfrak{B}$

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^1) \text{ s.t. } (3.2) \text{ are satisfied}\}.$$

It can be shown that $\mathfrak{B}$ belongs to $\mathcal{L}_2^{\mathtt{w}}$, in other words, it can be described by a set of kernel equations such as (3.1) (see [17]).

In the following we need the notion of (weakly) *autonomous* 2D-behavior. In order to formalize such a concept we need to define the characteristic ideal and characteristic variety associated with a kernel representation (3.1). Let $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \xi_2]$; its *characteristic ideal* is the ideal of $\mathbb{R}[\xi_1,\xi_2]$ generated by the determinants of all $\mathtt{w}\times\mathtt{w}$ minors of R, and the *characteristic variety* is the set of solutions common to all polynomials in the ideal. A behavior represented in kernel form by (3.1) is (weakly) *autonomous* if its characteristic ideal is not the zero ideal, or equivalently, if its characteristic variety is not all of $\mathbb{C}^2$. The characteristic variety is finite iff the behavior is finite-dimensional, i.e., it consists only of polynomial vector-exponential trajectories.

Finally, we introduce the notion of *dual* of a linear differential behavior. We denote by $\mathfrak{D}(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}})$ the set of infinitely differentiable trajectories from $\mathbb{R}^2$ to $\mathbb{C}^{\mathtt{w}}$ with compact support. Let $J \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be an involution, i.e., $J^2 = I_{\mathtt{w}}$; given a controllable behavior $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$, we define its J-*dual* as

$$\mathfrak{B}^{\perp_J} := \left\{ w' \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{C}^{\mathtt{w}}) \,\middle|\, \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w'^* J w \; dt_1 dt_2 = 0 \right.$$
$$\left. \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^2, \mathbb{R}^{\mathtt{w}}) \right\}. \tag{3.3}$$

Using an integration-by-parts argument it can be shown (see also [16, sect. 5]) that if $\mathfrak{B} = \ker\ R(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}) = \text{im}\ M(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2})$, then

$$\mathfrak{B}^{\perp_J} = \ker\ M\left(-\frac{\partial}{\partial t_1}, -\frac{\partial}{\partial t_2}\right)^\top J = \text{im}\ JR\left(-\frac{\partial}{\partial t_1}, -\frac{\partial}{\partial t_2}\right)^\top . \tag{3.4}$$

If $J = I$, we denote $\mathfrak{B}^{\perp_J}$ by $\mathfrak{B}^\perp$.

3.2. Two-dimensional bilinear differential forms. In order to simplify the notation, define the vector $\mathbf{t} := (t_1, t_2)$, the multi-indices $\mathbf{k} := (k_1, k_2)$ and $\mathbf{l} := (l_1, l_2)$, and the notation $\zeta := (\zeta_1, \zeta_2)$ and $\eta := (\eta_1, \eta_2)$. Thus $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ denotes the ring $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ of real polynomial $\mathtt{w}_1 \times \mathtt{w}_2$ matrices in the four indeterminates ζ_i and η_i, $i=1,2$, and $\zeta^{\mathbf{k}}\eta^{\mathbf{l}} = \zeta_1^{k_1}\zeta_2^{k_2}\eta_1^{l_1}\eta_2^{l_2}$.

An element of $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ is of the form $\Phi(\zeta,\eta)=\sum_{\mathbf{k},\mathbf{l}}\Phi_{\mathbf{k},\mathbf{l}}\zeta^{\mathbf{k}}\eta^{\mathbf{l}}$, where $\Phi_{\mathbf{k},\mathbf{l}}\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$; the sum ranges over all nonnegative multindices $\mathbf{k}$ and $\mathbf{l}$ and is assumed to be finite. Such a matrix induces a *bilinear differential form* (*BDF*) L_Φ

$$\begin{aligned} L_\Phi:\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_1})\times\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_2}) &\longrightarrow \mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C})\ ,\\ (v,w)&\longrightarrow \sum_{\mathbf{k},\mathbf{l}}\left(\frac{\partial^{\mathbf{k}}v}{\partial\mathbf{t}^{\mathbf{k}}}\right)^*\Phi_{\mathbf{k},\mathbf{l}}\frac{\partial^{\mathbf{l}}w}{\partial\mathbf{t}^{\mathbf{l}}}\ ,\end{aligned}$$

where the $\mathbf{k}$th derivatives $\frac{\partial^{\mathbf{k}}}{\partial\mathbf{t}^{\mathbf{k}}}$ and $\frac{\partial^{\mathbf{l}}}{\partial\mathbf{t}^{\mathbf{l}}}$ are defined by $\frac{\partial^{\mathbf{k}}}{\partial\mathbf{t}^{\mathbf{k}}}:=\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}}$, $\frac{\partial^{\mathbf{l}}}{\partial\mathbf{t}^{\mathbf{l}}}:=\frac{\partial^{l_1+l_2}}{\partial t_1^{l_1}\partial t_2^{l_2}}$.

Given $\mathfrak{B}_i\in\mathcal{L}_2^{\mathtt{w}}$, $i=1,2$, two BDFs L_{Φ_i}, $i=1,2$ are *equivalent along* $\mathfrak{B}_1\times\mathfrak{B}_2$, denoted by $L_{\Phi_1}\overset{\mathfrak{B}_1\times\mathfrak{B}_2}{=}L_{\Phi_2}$ or by $\Phi_1\overset{\mathfrak{B}_1\times\mathfrak{B}_2}{=}\Phi_2$, if $L_{\Phi_1}(v,w)=L_{\Phi_2}(v,w)$ for all $(v,w)\in\mathfrak{B}_1\times\mathfrak{B}_2$. In the following result we characterize equivalence of BDFs along behaviors in terms of properties of the associated polynomial matrices.

PROPOSITION 3.1. *Let* $\mathfrak{B}_1=\ker\ R_1(\frac{\partial}{\partial\mathbf{t}})\in\mathcal{L}_2^{\mathtt{w}_1}$, $\mathfrak{B}_2=\ker\ R_2(\frac{\partial}{\partial\mathbf{t}})\in\mathcal{L}_2^{\mathtt{w}_2}$, *and let* $\Phi_1\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $\Phi_2\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2]$.

$L_{\Phi_1}\overset{\mathfrak{B}_1\times\mathfrak{B}_2}{=}L_{\Phi_2}$ *if and only if there exist* $Y_i\in\mathbb{R}^{\bullet\times\bullet}[\xi_1,\xi_2]$, $i=1,2$, *such that*

$$\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)=\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)+R_1(\zeta_1,\zeta_2)^\top Y_1(\eta_1,\eta_2)+Y_2(\zeta_1,\zeta_2)^\top R_2(\eta_1,\eta_2)\ .$$

Proof. The proof of sufficiency is straightforward. To prove necessity, an argument analogous to that of [10, Prop. 10] can be used. ∎

In the following we often differentiate a BDF with respect to one of the independent variables, i.e., from L_Φ we define for $i=1,2$

$$\begin{aligned}\frac{\partial}{\partial t_i}L_\Phi:\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_1})\times\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_2})&\longrightarrow\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C})\ ,\\ (v,w)&\longrightarrow\frac{\partial}{\partial t_i}L_\Phi(v,w)\ .\end{aligned}$$

It is easy to see that the partial derivative of a BDF is also a BDF. Using Leibniz's rule for the expression $\frac{\partial}{\partial t_i}(\sum_{\mathbf{k},\mathbf{l}}(\frac{\partial^{\mathbf{k}}v}{\partial\mathbf{t}^{\mathbf{k}}})^*\Phi_{\mathbf{k},\mathbf{l}}\frac{\partial^{\mathbf{l}}w}{\partial\mathbf{t}^{\mathbf{l}}})$, it can be verified in a straightforward way that the polynomial matrix representing $\frac{\partial}{\partial t_i}L_\Phi$ is

$$(\zeta_i+\eta_i)\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2)\ ,i=1,2\ . \tag{3.5}$$

We also consider vectors $\Psi\in(\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2])^2$, i.e., $\Psi=(\Psi_1^\top,\Psi_2^\top)^\top$ with $\Psi_i\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $i=1,2$. Ψ induces a *vector of BDFs* (*VBDFs*) defined by

$$\begin{aligned}L_\Psi:\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_1})\times\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}^{\mathtt{w}_2})&\longrightarrow(\mathfrak{C}^\infty(\mathbb{R}^2,\mathbb{C}))^2,\\ L_\Psi(v,w)&\longrightarrow\ \mathrm{col}\left(\sum_{\mathbf{k},\mathbf{l}}\left(\frac{\partial^{\mathbf{k}}v}{\partial\mathbf{t}^{\mathbf{k}}}\right)^*\Psi_{i,\mathbf{k},\mathbf{l}}\frac{\partial^{\mathbf{l}}w}{\partial\mathbf{t}^{\mathbf{l}}}\right)_{i=1,2}\ ,\end{aligned}$$

where $\Psi_{i,\mathbf{k},\mathbf{l}}$ is the $(\mathbf{k},\mathbf{l})$-coefficient of the ith component of Ψ.

Finally, we introduce the notion of *divergence* of a VBDFs, the counterpart of the derivative of a BDF in the 1D case. Given a VBDFs $L_\Psi=\mathrm{col}\ (L_{\Psi_i})_{i=1,2}$, we define its divergence as the BDF defined by

$$(\mathrm{div}\ L_\Psi)(w_1,w_2):=\left(\frac{\partial}{\partial t_1}L_{\Psi_1}\right)(w_1,w_2)+\left(\frac{\partial}{\partial t_2}L_{\Psi_2}\right)(w_1,w_2) \tag{3.6}$$

for all infinitely differentiable trajectories w_1, w_2. In terms of the 4-variable polynomial matrices associated with the BDFs, the relationship between a VBDF and its divergence is expressed as (see [18, Thm. 4])

$$\begin{aligned} \text{div } \operatorname{col}(\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2),\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)) = {} & (\zeta_1+\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) \\ & + (\zeta_2+\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) \, . \end{aligned} \tag{3.7}$$

4. State representations of the MPUM. We recall the following definition from [32, p. 1160].

DEFINITION 4.1. *A state-representation* (2.2) *is* observable *if for every* $w \in \mathfrak{B}$ *there exists a unique* $x_0 \in \mathbb{R}^n$ *such that* (w, x) *satisfies* (2.2) $\Longrightarrow x(0,0) = x_0$.

Using the fact that $\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial {t_2}^{k_2}} w = C A_1^{k_1} A_2^{k_2} x$ for all $k_1, k_2 \in \mathbb{N}$, it can be verified in a straightforward way that a representation (2.2) is observable if and only if $\bigcap_{(k_1,k_2)\in\mathbb{N}^2} \ker\ C A_1^{k_1} A_2^{k_2} = \{0\}$. Any finite-dimensional behavior $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ admits an observable state representation (2.2); see [32, Thm. 4, p. 1160].

Our approach is based on the analysis of two infinite matrices computed from the data (2.1), which we now introduce. Given $\{w_i\}_{i=1,\dots,N}$ and $(k_1, k_2) \in \mathbb{N}^2$, we first define the matrix of the (k_1, k_2)th derivative of the data trajectories by

$$\mathcal{H}_{k_1,k_2} := \begin{bmatrix} \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} w_1 & \dots & \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} w_N \end{bmatrix} .$$

Now define the *matrix of jets* by

$$\mathcal{H} := \begin{bmatrix} \mathcal{H}_{0,0} & \mathcal{H}_{1,0} & \mathcal{H}_{0,1} & \mathcal{H}_{2,0} & \mathcal{H}_{1,1} & \mathcal{H}_{0,2} & \dots \\ \mathcal{H}_{1,0} & \mathcal{H}_{2,0} & \mathcal{H}_{1,1} & \mathcal{H}_{3,0} & \mathcal{H}_{2,1} & \mathcal{H}_{1,2} & \dots \\ \mathcal{H}_{0,1} & \mathcal{H}_{1,1} & \mathcal{H}_{0,2} & \mathcal{H}_{2,1} & \mathcal{H}_{1,2} & \mathcal{H}_{0,3} & \dots \\ \mathcal{H}_{2,0} & \mathcal{H}_{3,0} & \mathcal{H}_{2,1} & \mathcal{H}_{4,0} & \mathcal{H}_{3,1} & \mathcal{H}_{2,2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \tag{4.1}$$

The matrix $\mathcal{H}(0,0)$ of jets at $(0,0)$ is defined by

$$\mathcal{H}_{k_1,k_2}(0,0) := \begin{bmatrix} \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} w_1(0,0) & \dots & \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} w_N(0,0) \end{bmatrix}$$

and

$$\mathcal{H}(0,0) := \begin{bmatrix} \mathcal{H}_{0,0}(0,0) & \mathcal{H}_{1,0}(0,0) & \mathcal{H}_{0,1}(0,0) & \dots \\ \mathcal{H}_{1,0}(0,0) & \mathcal{H}_{2,0}(0,0) & \mathcal{H}_{1,1}(0,0) & \dots \\ \mathcal{H}_{0,1}(0,0) & \mathcal{H}_{1,1}(0,0) & \mathcal{H}_{0,2}(0,0) & \dots \\ \mathcal{H}_{2,0}(0,0) & \mathcal{H}_{3,0}(0,0) & \mathcal{H}_{2,1}(0,0) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \tag{4.2}$$

The following theorem is the main result of this section.

THEOREM 4.2. *Let* (2.1) *be given, and define* $\mathcal{H}$ *and* $\mathcal{H}(0,0)$ *by* (4.1) *and* (4.2), *respectively. Then rank*$(\mathcal{H}) = \mathtt{n} = \dim\ \mathfrak{B}^*$, *and rank*$(\mathcal{H}(0,0)) = \mathtt{n} = \dim\ \mathfrak{B}^*$. *Moreover, let* $\mathcal{H} = SX$ *be such that* $S \in \mathbb{C}^{\infty\times\mathtt{n}}$ *and* $X \in \mathbb{C}^{\mathtt{n}\times N}$, *and denote the ith column of* X *by* $\overline{x}_i$, $i = 1, \dots, N$. *There exists a minimal state representation* (2.2) *of* $\mathfrak{B}^*$ *with* x_i *the state trajectory corresponding to* w_i, $i = 1, \dots, N$, *such that* $x_i(0,0) = \overline{x}_i$, $i = 1, \dots, N$.

Proof. Recall that $\mathfrak{B}^* = \text{lin span}\,\{\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} w_i \mid (k_1,k_2) \in \mathbb{N}^2\ ,\ i = 1,\dots,N\}$. Consequently, $\mathfrak{B}^*$ coincides with the image of the submatrix $\mathcal{H}'$ of $\mathcal{H}$ consisting of its first $\mathtt{w}$ rows, and $\mathtt{n} = \dim\ \mathfrak{B}^* = \text{rank}\ \mathcal{H}'$. Now choose a set of $\mathtt{n}$ linearly independent columns of $\mathcal{H}'$ that generate im $\mathcal{H}'$, and consider the corresponding $\mathtt{n}$ columns of $\mathcal{H}$. The linearity of the operation of partial differentiation implies that any other column of $\mathcal{H}$ is linearly dependent on the selected set. Consequently dim $\mathcal{H} = \mathtt{n}$.

To prove the second statement, let (2.2) be a minimal state representation of $\mathfrak{B}$, and denote by x_i the state trajectory associated with w_i in such state representation. Recall that $\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2{}^{k_2}} w_i = CA_1^{k_1}A_2^{k_2}x_i$ for all $k_1,k_2 \in \mathbb{N}$, $i = 1,\dots,N$. Define

$$\mathcal{O} := \begin{bmatrix} C^\top & (CA_1)^\top & (CA_2)^\top & (CA_1^2)^\top & (CA_1A_2)^\top & \dots \end{bmatrix}^\top ,$$

where the powers of A_i, $i = 1,2$, in the jth block row of $\mathcal{O}$ are ordered in the same way as the partial derivatives in the jth block row of $\mathcal{H}$. Now define

$$\begin{aligned} \mathcal{X}_{k_1,k_2} &:= \begin{bmatrix} \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} x_1 & \dots & \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} x_N \end{bmatrix} ,\\ \mathcal{X}_{k_1,k_2}(0,0) &:= \begin{bmatrix} \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} x_1(0,0) & \dots & \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} x_N(0,0) \end{bmatrix} \end{aligned}$$

and observe that

$$\begin{aligned} \mathcal{H} &= \mathcal{O}\begin{bmatrix} \mathcal{X}_{0,0} & \mathcal{X}_{1,0} & \mathcal{X}_{0,1} & \mathcal{X}_{2,0} & \mathcal{X}_{1,1} & \dots \end{bmatrix}\\ &= \mathcal{O}\underbrace{\begin{bmatrix} \mathcal{X}_{0,0} & A_1\mathcal{X}_{0,0} & A_2\mathcal{X}_{0,0} & A_1^2\mathcal{X}_{0,0} & A_1A_2\mathcal{X}_{0,0} & \dots \end{bmatrix}}_{=:\mathcal{X}} ,\\ \mathcal{H}(0,0) &= \mathcal{O}\begin{bmatrix} \mathcal{X}_{0,0}(0,0) & \mathcal{X}_{1,0}(0,0) & \mathcal{X}_{0,1}(0,0) & \mathcal{X}_{2,0}(0,0) & \dots \end{bmatrix}\\ &= \mathcal{O}\underbrace{\begin{bmatrix} \mathcal{X}_{0,0}(0,0) & A_1\mathcal{X}_{0,0}(0,0) & A_2\mathcal{X}_{0,0}(0,0) & A_1^2\mathcal{X}_{0,0}(0,0) & \dots \end{bmatrix}}_{=:\mathcal{X}(0,0)} . \end{aligned}$$

Since the chosen state-representation is minimal, it is also observable, and consequently rank$(\mathcal{O}) = \mathtt{n}$. From this and statement (1) it follows that rank $\mathcal{X} = \mathtt{n}$. We now prove that rank $\mathcal{X}(0,0) = \mathtt{n}$; this will prove the second statement of the theorem.

Select $\mathtt{n}$ linearly independent columns of $\mathcal{H}'$, the submatrix consisting of the first $\mathtt{n}$ rows of $\mathcal{H}$. Consider the submatrix $\mathcal{X}'(0,0)$ consisting of the columns of $\mathcal{X}(0,0)$ corresponding to this selection of columns of $\mathcal{H}'$; denote its jth column by $\mathcal{X}_j'(0,0)$. Now assume by contradiction that there exist $\alpha_j \in \mathbb{C}$, $j = 1,\dots,\mathtt{n}$, not all zero, such that $\sum_{j=1}^{\mathtt{n}} \alpha_j\mathcal{X}_j'(0,0) = 0$. Then the trajectory of $\mathfrak{B}^*$ obtained by combining linearly the columns of $\mathcal{H}'$ with the coefficients α_j, $j = 1,\dots,\mathtt{n}$ is zero, since its corresponding state trajectory is $\sum_{j=1}^{\mathtt{n}} \alpha_j\mathcal{X}_j'(\cdot,\cdot)$ and it is zero at $(0,0)$. This leads to a contradiction: the chosen columns of $\mathcal{H}'$ were linearly independent by assumption.

To prove the last part of the theorem, consider that given any two factorizations $\mathcal{H} = \mathcal{O}\mathcal{X} = \mathcal{O}'\mathcal{X}'$, it holds that row span $\mathcal{H}$ = row span $\mathcal{X}$ = row span $\mathcal{X}'$, and consequently there exists a nonsingular matrix $T \in \mathbb{R}^{\bullet\times\bullet}$ such that $\mathcal{X}' = T\mathcal{X}$. Thus the columns of any matrix $\mathcal{X}$ obtained from a rank-revaling factorization of $\mathcal{H}$ are related by a nonsingular transformation to the vectors $x_i(0,0)$ corresponding to the value at $(0,0)$ of the state trajectories x_j corresponding to the w_j. ∎

It follows from Theorem 4.2 that any factorization $\mathcal{H}(0,0) = \mathcal{O}\mathcal{X}$ of the matrix $\mathcal{H}(0,0)$ such that rank $\mathcal{O}$ = rank $\mathcal{X}$ = rank $\mathcal{H}(0,0)$ yields a set of vectors associated with the values at $(0,0)$ of state trajectories associated with $\frac{\partial^{k_1+k_2}}{\partial^{k_1}t_1\partial^{k_2}t_2}w_i$. We call such a factorization a *rank-revealing factorization* of $\mathcal{H}(0,0)$. We now show how to exploit rank-revealing factorizations of $\mathcal{H}(0,0)$ to obtain a state-representation of $\mathfrak{B}^*$.

Let $\mathcal{H}(0,0) = \mathcal{O}\mathcal{X}$ be a rank-revealing factorization of $\mathcal{H}(0,0)$, and let X be any finite submatrix of $\mathcal{X}$ of rank $\mathtt{n} = \operatorname{rank}(\mathcal{X})$. It follows from Theorem 4.2 that there exist matrices A_i, $i = 1, 2$, and C of a minimal state-representation of $\mathfrak{B}^*$ such that each column of X is of the form $A_1^{k_1}A_2^{k_2}x_j(0,0)$ for some $k_1, k_2 \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$, where x_j is the state trajectory corresponding to w_j. Now denote by $\partial_1 X$, respectively, $\partial_2 X$, the $\mathtt{n} \times \mathtt{n}$ submatrix of $\mathcal{X}$ whose columns are $A_1^{k_1+1}A_2^{k_2}x_j(0,0)$, respectively, $A_1^{k_1}A_2^{k_2+1}x_j(0,0)$; we call these matrices the *shifts of X in the ith direction*.

PROPOSITION 4.3. *Let $\mathcal{H}(0,0) = \mathcal{O}\mathcal{X}$ be a rank-revealing factorization, and let X be any finite submatrix of $\mathcal{X}$ of rank $\mathtt{n} = \operatorname{rank}(\mathcal{X})$. Denote by $\partial_1 X$, respectively, $\partial_2 X$, the shifts of X in the first, respectively, second direction, and by $X^\dagger$ a right-inverse of X. Define $A_i := (\partial_i X)X^\dagger$, $i = 1, 2$, and the matrix C as that consisting of the first $\mathtt{w}$ rows of the matrix $\mathcal{O}$. Then (A_1, A_2, C) is a minimal state realization of $\mathfrak{B}^*$.*

Proof. It follows from Theorem 4.2 that given $\partial_i X$ and X, there exist matrices A_i of a realization of $\mathfrak{B}$ such that the equations $\partial_i X = A_i X$, $i = 1, 2$ are satisfied. Now use the assumption that X has full row rank $\mathtt{n}$ to conclude that $A_i = (\partial_i X)X^\dagger$, $i = 1, 2$. The last part of the proof follows in a straightforward way. $\square$

We now state an algorithm for computing a representation (2.2) of $\mathfrak{B}^*$.

Algorithm 1.
Input: Vector-exponential trajectories $\overline{w^i}e^{\lambda_1^i \cdot}e^{\lambda_2^i \cdot}$, $i = 1, \ldots, N$;
Output: Minimal representation (2.2) of $\mathfrak{B}^*$ for lin span $\{\overline{w^i}e^{\lambda_1^i \cdot}e^{\lambda_2^i \cdot}\}_{i=1,\ldots,N}$.
Compute $\mathcal{H}(0,0)$;
Compute a rank-revealing factorization $\mathcal{H}(0,0) = \mathcal{O}\mathcal{X}$;
Select submatrix X of $\mathcal{X}$ such that $\operatorname{rank}(X) = \operatorname{rank}(\mathcal{X})$;
Define $\partial_i X$ to be the ith shift of X, $i = 1, 2$;
Define $A_i := \partial_i X X^\dagger$, $i = 1, 2$;
Define C :=submatrix consisting of the first $\mathtt{w}$ rows and $\mathtt{n}$ columns of $\mathcal{H}(0,0)$.
Return A_1, A_2, and C.

We illustrate the application of Algorithm 1 with an example.

Example 1. Consider trajectories whose value at (t_1, t_2) is $w_1(t_1, t_2) := e^{2t_1}$, $w_2(t_1, t_2) := t_1 e^{3t_1}e^{5t_2}$. The matrix obtained from w_1, w_2 and their derivatives with columns and rows ordered in the total degree lexicographic ordering up to $(0, 2)$ is

$$\mathcal{H}'(0,0) = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 & 4 & 6 & 0 & 5 & 0 & 0 \\ 2 & 1 & 4 & 6 & 0 & 5 & 8 & 27 & 0 & 30 & 0 & 25 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 30 & 0 & 25 & 0 & 0 \\ 4 & 6 & 8 & 27 & 0 & 30 & 16 & 108 & 0 & 135 & 0 & 150 \\ 0 & 5 & 0 & 30 & 0 & 25 & 0 & 135 & 0 & 150 & 0 & 125 \\ 0 & 0 & 0 & 25 & 0 & 0 & 0 & 150 & 0 & 125 & 0 & 0 \end{bmatrix},$$

and it has rank 3. The rank does not increase adding columns and rows of $\mathcal{H}(0,0)$; consequently, the minimal dimension of the state space is 3. A rank-revealing factorization of $\mathcal{H}'(0,0)$ is $\mathcal{H}'(0,0) = \mathcal{O}'X'$, where

$$\mathcal{O}'^\top = \begin{bmatrix} -0.363 & -2.49 & -1.78 & -11.7 & -12.4 & -8.89 \\ -0.321 & 0.393 & -1.78 & 5.28 & 1.61 & -8.90 \\ 1.23 & 1.92 & 0.115 & 1.68 & -2.44 & 0.575 \end{bmatrix}^\top$$

and

$$X' := \begin{bmatrix} -0.138 & 0.191 & 0.823 \\ -0.356 & 0.355 & -0.0127 \\ -0.276 & 0.382 & 1.65 \\ -2.47 & -0.341 & -0.00574 \\ 0 & 0 & 0 \\ -1.78 & 1.77 & -0.0635 \\ -0.551 & 0.763 & 3.29 \\ -11.6 & -5.24 & 0.0798 \\ 0 & 0 & 0 \\ -12.4 & -1.70 & -0.0287 \\ 0 & 0 & 0 \\ -8.90 & 8.87 & -0.317 \end{bmatrix}^{\top} .$$

A full row-rank submatrix of X' is that consisting of the first, second, and fourth column. The corresponding $\partial_1 X'$ consists of the third, fourth, and eighth column of X'; $\partial_2 X'$ consists of columns 5, 6, and 10 of X'. Solving $\partial_i X = A_i X$, $i = 1, 2$, in the least-squares sense yields

$$A_1 = \begin{bmatrix} 4.97 & -1.95 & 0.948 \\ 1.97 & 1.04 & 0.553 \\ -0.0391 & 0.0158 & 1.99 \end{bmatrix} , \ A_2 = \begin{bmatrix} 4.99 & 0.0244 & 0.831 \\ 0.00733 & 4.97 & -1.15 \\ 0.0316 & -0.146 & 0.0391 \end{bmatrix} ,$$

while the first row of $\mathcal{O}'$ gives $C = \begin{bmatrix} -0.363 & -0.321 & 1.23 \end{bmatrix}$. The norm of $A_1 A_2 - A_2 A_1$ is of the order of 10^{-14}, suggesting that if the factorizations could be performed in infinite precision, the commutativity of state matrices for state-representations (2.2) of the MPUM would be verified. □

5. Roesser state models from i/o data.

This section is divided in three parts. In the first one we show that an inner product of external primal and dual trajectories is the divergence of a field whose components are the inner products of the first and second state variables of the primal and dual Roesser models. In section 5.2 we characterize zero-divergence fields, also along a pair of behaviors. In section 5.3 we show how to compute a Roesser model interpolating given vector-exponential trajectories on the basis of the decomposition of a constant matrix derived from the data in the sum of two lower-rank matrices. Finally, in section 5.4 we illustrate our procedure with a numerical example and comment on several issues.

5.1. Duality and divergence of fields of state variables.

We associate to a Roesser representation (2.3) its *dual* one, defined by the equations

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial t_1} x_1' \\ \frac{\partial}{\partial t_2} x_2' \end{bmatrix} &= -A^{\top} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} C_1^{\top} \\ C_2^{\top} \end{bmatrix} u' , \\ y' &= \begin{bmatrix} B_1^{\top} & B_2^{\top} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} - D^{\top} u' . \end{aligned} \tag{5.1}$$

The adjective "dual" is justified by the following result.

PROPOSITION 5.1. *Assume that the external behaviors* $\mathfrak{B}$*, respectively,* $\mathfrak{B}'$*, of* (2.3) *and* (5.1)*, respectively, are controllable. Then* $\mathfrak{B}' = \mathfrak{B}^{\perp}$ *is defined by* (3.3).

Proof. We show that the transfer functions $H'(\xi_1, \xi_2)$ of $\mathfrak{B}'$ and $H(\xi_1, \xi_2)$ of $\mathfrak{B}$ satisfy $H'(\xi_1, \xi_2) = -H(\xi_1, \xi_2)^\top$. This leads straightforwardly to the claim. Partition $A =: \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ accordingly to the partitions of x and x'; it is a matter of verification using 2D-Laplace transforms to check that

$$H(\xi_1, \xi_2) = C \begin{bmatrix} \xi_1 I_{\mathtt{n}_1} - A_{11} & -A_{12} \\ -A_{21} & -\xi_2 I_{\mathtt{n}_2} - A_{22} \end{bmatrix}^{-1} B + D \,,$$

$$H'(\xi_1, \xi_2) = B^\top \begin{bmatrix} \xi_1 I_{\mathtt{n}_1} + A_{11}^\top & A_{21}^\top \\ A_{12}^\top & \xi_2 I_{\mathtt{n}_2} + A_{22}^\top \end{bmatrix}^{-1} C^\top - D^\top \,,$$

from which it follows that $H'(\xi_1, \xi_2) = -H(-\xi_1, -\xi_2)^\top$. Now let $N(\xi_1, \xi_2) D(\xi_1, \xi_2)^{-1}$ be a right-coprime factorization of $H(\xi_1, \xi_2)$; then $-D(-\xi_1, -\xi_2)^{-\top} N(-\xi_1, -\xi_2)^\top$ is a left-coprime factorization of $H'(\xi_1, \xi_2)$. Consequently the external behavior of (2.3) is im $\begin{bmatrix} D(\frac{\partial}{\partial \mathrm{t}_1}, \frac{\partial}{\partial \mathrm{t}_2}) \\ N(\frac{\partial}{\partial \mathrm{t}_1}, \frac{\partial}{\partial \mathrm{t}_2}) \end{bmatrix}$ and that of (5.1) is ker $\begin{bmatrix} N(-\frac{\partial}{\partial \mathrm{t}_1}, -\frac{\partial}{\partial \mathrm{t}_2})^\top & -D(-\frac{\partial}{\partial \mathrm{t}_1}, -\frac{\partial}{\partial \mathrm{t}_2})^\top \end{bmatrix}$. To conclude the proof use (3.4). ∎

In the computation of i/s/o representations from vector-exponential data, an important role is played by the following result.

PROPOSITION 5.2. *Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ be controllable, and let $\mathfrak{B} =$ ker $R(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}) =$ im $M(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2})$. Define $\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) := R(-\zeta_1, -\zeta_2) M(\eta_1, \eta_2) \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$. Then there exist $\Psi_1, \Psi_2 \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that*

$$\begin{aligned} \Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) &= \operatorname{div}\left(\operatorname{col}(\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2), \Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2))\right) \\ &= (\zeta_1 + \eta_1) \Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (\zeta_2 + \eta_2) \Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) \,. \end{aligned}$$

Proof. Recall that $R(\xi_1, \xi_2) M(\xi_1, \xi_2) = 0$, and apply [18, Thm. 4, p. 1411]. ∎

From Propositions 5.1 and 5.2 it follows that there exists a vector $\operatorname{col}(L_{\Psi_1}, L_{\Psi_2})$ of bilinear differential forms acting on the external variables of the primal and dual system, such that for every $w \in \mathfrak{B}$ and $w' \in \mathfrak{B}^\perp$ the following equality holds:

$$w'^* w = \operatorname{div}\left(L_{\Psi_1}(w, w'), L_{\Psi_2}(w, w')\right) \,. \tag{5.2}$$

We show that $w'^* w$ is the divergence of a VBDF's acting, respectively, on the *first* and *second state variables* associated to w and w' in the primal and the dual system.

THEOREM 5.3. *Let $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ with state representations* (2.3) *and* (5.1), *respectively. Let $w = \operatorname{col}(u, y) \in \mathfrak{B}$ and $w' = \operatorname{col}(u', y') \in \mathfrak{B}'$ with associated state trajectories $x = \operatorname{col}(x_1, x_2)$ and $x' = \operatorname{col}(x_1', x_2')$, respectively. Then*

$$\begin{bmatrix} u^* & y^* \end{bmatrix} \begin{bmatrix} 0_{\mathtt{m} \times \mathtt{p}} & I_{\mathtt{m}} \\ I_{\mathtt{p}} & 0_{\mathtt{p} \times \mathtt{m}} \end{bmatrix} \begin{bmatrix} u' \\ y' \end{bmatrix} = \operatorname{div}\left(x_1^* x_1', x_2^* x_2'\right) = \frac{\partial}{\partial \mathrm{t}_1}\left(x_1^* x_1'\right) + \frac{\partial}{\partial \mathrm{t}_2}\left(x_2^* x_2'\right) \,. \tag{5.3}$$

Proof. The claim follows from the following chain of equalities:

$$\begin{aligned} y^* u' + u^* y' &= \left(x^* C^\top + u^* D^\top\right) u' + u^* \left(B^\top x' - D^\top u'\right) = x^* \left(C^\top u'\right) + \left(u^* B^\top\right) x' \\ &= x^* \left(A^\top x' + \begin{bmatrix} \frac{\partial}{\partial \mathrm{t}_1} x_1' \\ \frac{\partial}{\partial \mathrm{t}_2} x_2' \end{bmatrix}\right) + \left(\begin{bmatrix} \frac{\partial}{\partial \mathrm{t}_1} x_1 \\ \frac{\partial}{\partial \mathrm{t}_2} x_2 \end{bmatrix} - A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)^* x' \,. \end{aligned}$$ ∎

Remark 1. If (w,x) and (w',x') are full trajectories of two 1D behaviors $\mathfrak{B}$, respectively, $\mathfrak{B}^{\perp}$, then $w'^* w = \frac{d}{dt}\left(x'^{\top} x\right)$ (see [30, Prop. 10.1, p. 1730]). Such a relation is at the basis of the 1D Loewner approach to rational interpolation; see [3]. Theorem 5.3 provides an analogous result in the 2D case. ☐

Remark 2. From Theorem 5.3 follows an alternative proof of Proposition 5.1. Indeed, for any pair consisting of a compact support trajectory $w = \mathrm{col}(u,y) \in \mathfrak{B}$ and $w' = \mathrm{col}(u',y') \in \mathfrak{B}'$ and associated state trajectories x, x', (5.3) implies that

$$\int\int_{-\infty}^{+\infty} u^* y' + y^* u' dt_1 dt_2 = \frac{\partial}{\partial \mathrm{t}_1}\left(x_1^* x_1'\right)|_{t_1=-\infty}^{t_1=+\infty} + \frac{\partial}{\partial \mathrm{t}_2}\left(x_2^* x_2'\right)|_{t_2=-\infty}^{t_2=+\infty} = 0 \,, \tag{5.4}$$

where the last equality is justified by the fact that since the external trajectories have compact support, x also has compact support. ☐

In the rest of the paper we assume that the data is purely vector-exponential, i.e., $L_1^i = 0 = L_2^i$, $i = 1,2$ in (2.1). For the moment we also assume that a set of N dual trajectories is known; we show in Remark 5 that such an assumption is of little import, since dual trajectories are readily computed from primal ones. Consequently, for the time being we assume that the following data is available:

$$\begin{aligned} w_i(\cdot,\cdot) &= \mathrm{col}(u,y)(\cdot,\cdot) = \begin{bmatrix} \overline{u_i} \\ \overline{y_i} \end{bmatrix} e^{\lambda_1^i \cdot} e^{\lambda_2^i \cdot} \in \mathfrak{B} \,,\ i = 1,\dots,N \,, \\ w_i'(\cdot,\cdot) &= \mathrm{col}(u',y')(\cdot,\cdot) = \begin{bmatrix} \overline{u_i'} \\ \overline{y_i'} \end{bmatrix} e^{\mu_1^i \cdot} e^{\mu_2^i \cdot} \in \mathfrak{B}^{\perp_J} \,,\ i = 1,\dots,N \,. \end{aligned} \tag{5.5}$$

To such trajectories correspond vector-exponential state trajectories

$$\overline{x_i} e^{\lambda_1^i \cdot} e^{\lambda_2^i \cdot} \,,\ \overline{x_j}' e^{\mu_1^i \cdot} e^{\mu_2^i \cdot} \tag{5.6}$$

(where $\overline{x_i}', \overline{x_i} \in \mathbb{C}^{\mathtt{n}}$), $i = 1,\dots,N'$, $j = 1,\dots,N$, satisfying (5.1) and (2.3), respectively. We partition $\overline{x_i}' =: \mathrm{col}(\overline{x_{i,1}}', \overline{x_{i,2}}')$ and $\overline{x_i} =: \mathrm{col}(\overline{x_{i,1}}, \overline{x_{i,2}})$ according to the partition of the state trajectories in (5.1) and (2.3).

Define from (5.5), (5.6) the matrices

$$\begin{aligned} &L := \begin{bmatrix} w_1' & \dots & w_N' \end{bmatrix} \in \mathbb{C}^{\mathtt{w}\times N} \,,\ R := \begin{bmatrix} w_1 & \dots & w_N \end{bmatrix} \in \mathbb{C}^{\mathtt{w}\times N} \,, \\ &\Lambda_i := \mathrm{diag}(\lambda_i^k)_{k=1,\dots,N} \,,\ M_i := \mathrm{diag}(\mu_i^k)_{k=1,\dots,N} \,,\ i = 1,2 \,, \\ &X' := \begin{bmatrix} \overline{x_1}' & \dots & \overline{x_{N'}} \end{bmatrix} =: \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \,,\ X := \begin{bmatrix} \overline{x_1} & \dots & \overline{x_N} \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times N} \,. \end{aligned} \tag{5.7}$$

The following result, a straightforward consequence of Theorem 5.3, establishes the connection between matrices computed from the external data and matrices computed from the internal (i.e., state) ones.

PROPOSITION 5.4. *Define the matrices L, R, Λ_i, M_i, X', X by* (5.7). *Then*

$$L^* J R = M_1^* X_1'^* X_1 + X_1'^* X_1 \Lambda_1 + M_2^* X_2'^* X_2 + X_2'^* X_2 \Lambda_2 \,. \tag{5.8}$$

Proof. The claim follows in a straightforward way considering the value at $(0,0)$ of (5.3) on the external data and their associated state trajectories. ☐

We now give sufficient conditions under which the matrices X and X' defined in (5.6) have rank equal to the dimension of the state variables x_i, x_i', $i = 1,2$.

PROPOSITION 5.5. *Let $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ be controllable. Let* (5.5) *be given, and denote by $x_i(\cdot,\cdot)$, respectively, $x'_j(\cdot,\cdot)$, a state trajectory corresponding to $w_i(\cdot,\cdot)$, respectively, $w'_j(\cdot,\cdot)$, in a Roesser representation* (2.3)*, respectively,* (5.1)*.*

Assume that $N > \mathtt{n}_1 + \mathtt{n}_2$ and that there exist $\mathtt{n} = \mathtt{n}_1 + \mathtt{n}_2$ linearly independent trajectories among those in $\{w_i\}_{i=1,\ldots,N}$ and $\{w'_i\}_{i=1,\ldots,N}$, respectively. Then

$$\operatorname{rank}\begin{bmatrix} x_{i,1}(0) & \ldots & x_{i,N}(0)\end{bmatrix} = \mathtt{n}_i = \operatorname{rank}\begin{bmatrix} x'_{i,1}(0) & \ldots & x'_{i,N}(0)\end{bmatrix}\ ,\ i = 1, 2\ .$$

Proof. We prove the first and third equalities; the other two follow in an analogous manner. Assume by contradiction that $\operatorname{rank} \overline{X} := \operatorname{rank}\begin{bmatrix} x_{1,1}(0) \ldots x_{1,N}(0) \\ x_{2,1}(0) \ldots x_{2,N}(0)\end{bmatrix} < \mathtt{n}_1 + \mathtt{n}_2$. Reordering the trajectories w_i if needed, we can assume that the first $\mathtt{n}$ of them are linearly independent. Since $\operatorname{rank} \overline{X} < \mathtt{n}$, the submatrix $\overline{X}'$ of $\overline{X}$ consisting of its first $\mathtt{n}$ columns is such that there exist $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, \mathtt{n}$, not all zero, such that $\overline{X}' \operatorname{col}(\alpha_i)_{i=1,\ldots,\mathtt{n}_1+\mathtt{n}_2} = 0$. The MPUM for the set $\{\operatorname{col}(w_i, x_i)\}_{i=1,\ldots,\mathtt{n}_1+\mathtt{n}_2}$ is autonomous and finite-dimensional; moreover, since such behavior is a subset of the set of full (external, state) trajectories of a state representation of $\mathfrak{B}$, x is a state variable also for it. Now define $\widehat{w}(\cdot,\cdot) := \sum_{k=1}^{\mathtt{n}_1+\mathtt{n}_2} \alpha_k \overline{w_k} e^{\lambda_1^k \cdot} e^{\lambda_2^k \cdot}$; its associated state trajectory is $\widehat{x}(\cdot,\cdot) := \sum_{k=1}^{\mathtt{n}_1+\mathtt{n}_2} \alpha_k x_k(0,0) e^{\lambda_1^k \cdot} e^{\lambda_2^k \cdot}$. Since $\overline{X}' \operatorname{col}(\alpha_i)_{i=1,\ldots,\mathtt{n}_1+\mathtt{n}_2} = 0$, the value at $(0,0)$ of such state trajectory is zero. Given that the MPUM is autonomous, this implies that $\hat{w}$ is also zero. This however is in contradiction with the linear independence of the first $\mathtt{n}$ external trajectories and the assumption that not all α_i's are equal to zero. Consequently $\overline{X}$ has rank $\mathtt{n}_1 + \mathtt{n}_2$. This is readily seen to imply that $\operatorname{rank}\begin{bmatrix} x_{1,1}(0) & \ldots & x_{1,N}(0)\end{bmatrix} = \mathtt{n}_1$ and $\operatorname{rank}\begin{bmatrix} x_{2,1}(0) & \ldots & x_{2,N}(0)\end{bmatrix} = \mathtt{n}_2$. ◻

Remark 3. A sufficient condition for the external vector-exponential trajectories to be linearly independent is that $(\lambda_1^k, \lambda_2^k) \neq (\lambda_1^j, \lambda_2^j)$ for $j \neq k$, $j, k = 1, \ldots, N$, and $(\mu_1^i, \mu_2^i) \neq (\mu_1^\ell, \mu_2^\ell)$, for $i \neq \ell$, $i, \ell = 1, \ldots, N$. ◻

5.2. Zero-divergence fields and their characterization. Given two controllable behaviors $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ described by (2.3) and (5.1), the $\mathtt{n}_i \in \mathbb{N}$, matrices $X_i \in \mathbb{R}^{\mathtt{n}_i \times N}$, $X'_i \in \mathbb{R}^{\mathtt{n}_i \times N}$, $i = 1, 2$, defined by (5.7) satisfy (5.8). If such X_i, X'_i can be computed from the left-hand side of (5.8) and a "sufficiently informative" set of data is available, then matrices (A, B, C, D) of a Roesser model for the primal system can be computed solving a system of linear equations. However, given $L^* J R$ the solutions X_i, X'_i, $i = 1, 2$ to (5.8) are nonunique, since the homogeneous matrix equation in $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}^{N \times N}$

$$0 = M_1^* \mathcal{X}_1 + \mathcal{X}_1 \Lambda_1 + M_2^* \mathcal{X}_2 + \mathcal{X}_2 \Lambda_2 \tag{5.9}$$

has nonzero solutions $\mathcal{X}_i$, $i = 1, 2$. Such nonuniqueness arises since the divergence operator appearing on the right-hand side of (5.3), from which (5.8) derives, is *non-invertible*: given a 2D function f, there are many fields $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ such that $\nabla F = f$. See also [29, sect. 27-4] on the issues arising in Maxwell's equations from the noninjectivity of the divergence operator.

The following result is a characterization of zero-divergence fields in terms of properties of the corresponding polynomial matrices.

PROPOSITION 5.6. *Let $\Psi_i \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $i = 1, 2$. The following three statements are equivalent:*

1. $\operatorname{div}\ \operatorname{col}(L_{\Psi_1}, L_{\Psi_2}) = 0$.
2. $(\zeta_1 + \eta_1)\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (\zeta_2 + \eta_2)\Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) = 0$.

3. *There exists* $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ *such that*

$$\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) = (\zeta_2+\eta_2)\Psi(\zeta_1,\zeta_2,\eta_1,\eta_2),$$
$$\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) = -(\zeta_1+\eta_1)\Psi(\zeta_1,\zeta_2,\eta_1,\eta_2)\ .$$

Moreover, let $\mathfrak{B}_i = \ker\ R_i\left(\frac{\partial}{\partial \mathfrak{t}}\right) \in \mathcal{L}_2^{\mathtt{w}}$, $i=1,2$. *The following three statements are equivalent:*

4. div $\mathrm{col}(L_{\Psi_1}, L_{\Psi_2}) \overset{\mathfrak{B}_1\times\mathfrak{B}_2}{=} 0$.
5. $(\zeta_1+\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) + (\zeta_2+\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) \overset{\mathfrak{B}_1\times\mathfrak{B}_2}{=} 0$.
6. *There exist* $Y_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $i=1,2$, *such that*

$$\begin{aligned}&(\zeta_1+\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) + (\zeta_2+\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\,,\\ &\quad = R_1(\zeta_1,\zeta_2)^\top Y_1(\zeta_1,\zeta_2,\eta_1,\eta_2) + Y_2(\zeta_1,\zeta_2,\eta_1,\eta_2)^\top R_2(\eta_1,\eta_2).\end{aligned}$$

Proof. The equivalence of statements (1) and (2) follows from (3.7).

That (3) $\Longrightarrow$ (2) holds is a matter of straightforward verification.

The implication (2) $\Longrightarrow$ (3) follows observing that if $(\zeta_1+\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) = -(\zeta_2+\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)$, then Ψ_1 is divisible by $(\zeta_2+\eta_2)$, and Ψ_2 by $(\zeta_1+\eta_1)$. Consequently, there exist $\Psi'_j(\zeta_1,\zeta_2,\eta_1,\eta_2)$, $j=1,2$, such that $\Psi_j(\zeta_1,\zeta_2,\eta_1,\eta_2) = (\zeta_i+\eta_i)\Psi'_j(\zeta_1,\zeta_2,\eta_1,\eta_2)$, $i,j=1,2$, $i\neq j$. Statement (3) follows readily from such equality.

To prove the second part of the claim, the equivalence of (4) and (5) follows from (3.7). The equivalence of (5) and (6) follows from (3.7) and Proposition 3.1. ∎

The identifiability issues raised by the noninvertibility of the divergence operator will be considered elsewhere (see Proposition 5.5 below for a preliminary result); in the next section we present a procedure to compute Roesser models for the data (5.5).

5.3. Computing Roesser unfalsified models. The following result follows from Proposition 5.4.

THEOREM 5.7. *Let* $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ *be controllable and quarter-plane causal. Let data* (5.5) *be given and define* L, R, Λ_i, M_i, $i=1,2$, *by* (5.7). *Define*

$$\begin{aligned}&U := \begin{bmatrix}\overline{u_1} & \ldots & \overline{u_N}\end{bmatrix} \in \mathbb{C}^{\mathtt{m}\times N}\ ,\ Y := \begin{bmatrix}\overline{y_1} & \ldots & \overline{y_N}\end{bmatrix} \in \mathbb{C}^{\mathtt{p}\times N}\ ,\\ &U' := \begin{bmatrix}\overline{u'_1} & \ldots & \overline{u'_N}\end{bmatrix} \in \mathbb{C}^{\mathtt{p}\times N}\ ,\ Y' := \begin{bmatrix}\overline{y'_1} & \ldots & \overline{y'_N}\end{bmatrix} \in \mathbb{C}^{\mathtt{m}\times N}\ .\end{aligned} \tag{5.10}$$

There exist $\mathtt{n}_i \in \mathbb{N}$, *matrices* $X_i, X'_i \in \mathbb{C}^{\mathtt{n}_i\times N}$ $i=1,2$, *such that* (5.8) *holds. Moreover, there exist* $A_{ij} \in \mathbb{R}^{\mathtt{n}_i\times\mathtt{n}_j}$, $i=1,2$, $C_i \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}_i}$, $B_i \in \mathbb{R}^{\mathtt{n}_i\times\mathtt{m}}$, $i=1,2$, *such that the following equations hold:*

$$\begin{aligned}\begin{bmatrix}X_1\Lambda_1\\ X_2\Lambda_2\end{bmatrix} &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix} U\ ,\\ Y &= \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} + DU\ ,\\ \begin{bmatrix}X'_1 M_1\\ X'_2 M_2\end{bmatrix} &= -\begin{bmatrix}A_{11}^\top & A_{21}^\top\\ A_{12}^\top & A_{22}^\top\end{bmatrix}\begin{bmatrix}X'_1\\ X'_2\end{bmatrix} + \begin{bmatrix}C_1^\top\\ C_2^\top\end{bmatrix} U'\ ,\\ Y' &= \begin{bmatrix}B_1^\top & B_2^\top\end{bmatrix}\begin{bmatrix}X'_1\\ X'_2\end{bmatrix} - D^\top U'\ .\end{aligned} \tag{5.11}$$

Given such matrices A_{ij}, B_i, C_i, $i,j=1,2$, (2.3) *and* (5.1) *define unfalsified Roesser models for the data* (5.5).

Proof. Since $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ are controllable, they admit Roesser state representations (2.3) and (5.1), respectively. Denote by $x_i = \mathrm{col}(x_{1,i}, x_{2,i})$, $x'_i = \mathrm{col}(x'_{1,i}, x'_{2,i})$, the state trajectories associated in such representations with w_i, respectively, w'_i, $i = 1, \ldots, N$. Now consider the value at $(0,0)$ of (2.3) and (5.1) with such external- and state trajectories. This argument proves the first part of the theorem and (5.11). The last part of the claim is straightforward. ◻

From Theorem 5.7 it follows that the crucial issue in computing unfalsified models for the primal data is finding matrices S_i, $i = 1, 2$, solving the Sylvester-type equation

$$L^* J R = M_1^* S_1 + S_1 \Lambda_1 + M_2^* S_2 + S_2 \Lambda_2 \ , \tag{5.12}$$

from which matrices X_i, X'_i, $i = 1, 2$, can be computed such that the first two equations in (5.11) are satisfied. The following result gives sufficient conditions on S_1, S_2 and the data for this to happen.

THEOREM 5.8. *Let $\mathfrak{B}, \mathfrak{B}^\perp \in \mathcal{L}_2^{\mathtt{w}}$ be controllable. Let data* (5.5) *be given and define L, R, Λ_i, M_i, $i = 1, 2$ by* (5.7) *and U, Y, U', Y' by* (5.10).

Assume that $\mathrm{im}\ Y'^* \cap \mathrm{im}\ U'^* = \{0\}$. *Assume also that $S_1, S_2 \in \mathbb{C}^{N \times N}$ solve* (5.12) *and moreover that*

1. $\mathrm{im}\ S_1 \cap \mathrm{im}\ S_2 = \{0\}$;
2. $\mathrm{im}\ \begin{bmatrix} S_1 & S_2 \end{bmatrix} \cap \mathrm{im}\ U'^* = \{0\}$.

Let $S_i = X_i'^* X_i$, $i = 1, 2$ be rank-revealing factorizations. There exist a left inverse $\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger$ of $\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}$ and $F \in \mathbb{C}^{\mathtt{p} \times N}$ such that

$$\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger U'^* = 0_{N \times \mathtt{p}} \text{ and } F \begin{bmatrix} Y'^* & U'^* \end{bmatrix} = \begin{bmatrix} 0_{\mathtt{p} \times \mathtt{m}} & I_{\mathtt{p}} \end{bmatrix} \ . \tag{5.13}$$

Let $\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger$ and F satisfy (5.13), *and define*

$$\begin{aligned} A &:= -\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger \begin{bmatrix} M_1^* X_1'^* & M_2^* X_2'^* \end{bmatrix} \ , \ B := \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger Y'^* \ , \\ C &:= F \left(I_N - \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix} \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger \right) \begin{bmatrix} M_1^* X_1'^* & M_2^* X_2'^* \end{bmatrix} \ , \\ D &:= F \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix} \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger Y'^* \ . \end{aligned} \tag{5.14}$$

Then A, B, C, D define an unfalsified Roesser model for the data.

Proof. From $S_i = X_i'^* X_i$, $i = 1, 2$, being rank-revealing factorizations we conclude that $\mathrm{im}\ \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \mathrm{im}\ \begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}$. From this and assumption (1) it follows that $\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}$ admits a left inverse. From assumption (2) conclude that such a left-inverse can be chosen satisfying the first equation in (5.13).

Now multiply both sides of (5.12) by such a left-inverse to conclude that

$$\begin{bmatrix} X_1 \Lambda_1 \\ X_2 \Lambda_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + BU \ , \tag{5.15}$$

where A and B are defined by the first two equations in (5.14). Use the assumption im $Y'^* \cap$ im $U'^* = \{0\}$ to conclude that an F exists such that the second equation in (5.13) holds. Multiply both sides of (5.12) by such F and use (5.15) to conclude that

$$Y = C \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + DU \ , \tag{5.16}$$

where C and D are defined by the last two equations in (5.14). The fact that A, B, C, and D define an unfalsified model for the primal data follows from (5.15) and (5.16). This concludes the proof of the theorem. ∎

Based on the results of Theorems 5.7 and 5.8, to compute a Roesser model for data (5.7) we can proceed as follows. Assume that the condition im $Y'^\top \cap$ im $U'^\top = \{0\}$ with Y', U' defined by (5.10) is satisfied. Beginning with $(\mathtt{n}_1, \mathtt{n}_2) := (1, 0)$ and following the total degree lexicographic ordering in $\mathbb{N} \times \mathbb{N}$, we check the existence of a solution $\begin{bmatrix} S_1 & S_2 \end{bmatrix}$ to (5.12) with rank $S_i = \mathtt{n}_i$, $i = 1, 2$, satisfying conditions (1)–(2) of Theorem 5.8. Such check can be performed as follows: let $\begin{bmatrix} \overline{S}_1 & \overline{S}_2 \end{bmatrix}$ be a solution of (5.12); note that a solution always exists, since the data belongs to a controllable model and consequently (5.3) is satisfied. Now define

$$\mathcal{G} := \{(G_1, G_2) \mid G_1, G_2 \text{ solve } (5.9)\} \ , \tag{5.17}$$

and note that since (5.9) is a linear matrix equation, a parametrization of $\mathcal{G}$ is straightforward to obtain. We can now check whether there exist $(G_1, G_2) \in \mathcal{G}$ such that $S_1 := \overline{S}_1 + G_1$, $S_2 := \overline{S}_2 + G_2$ satisfy conditions (1)–(2) of Theorem 5.8; a mixed symbolic-numerical method is illustrated in Example 2 below. If such G_1, G_2 exist, then rank-revealing factorizations of $\overline{S}_i + G_i$, $i = 1, 2$, together with (5.13)–(5.14) yield an unfalsified model. If they do not, we can update $(\mathtt{n}_1, \mathtt{n}_2)$ to the next element of $\mathbb{N} \times \mathbb{N}$ in the total degree lexicographic order and start over. The model identified in this way is also of *minimal complexity* (state dimension) $\mathtt{n}_1 + \mathtt{n}_2$ among the unfalsified Roesser models for the data satisfying the conditions of Theorem 5.8.

5.4. Example and comments. We give an example of the application of our procedure and some comments and remarks addressing important issues.

Example 2. Consider the Roesser model associated with the matrices $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, and $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$, corresponding to the transfer function $G(\xi_1, \xi_2) = \frac{2\xi_1 + 2\xi_2 - 1}{\xi_1\xi_2 - 2\xi_1 - \xi_2 + 1}$, and the image and kernel representations

$$M(\xi_1, \xi_2) = \begin{bmatrix} \xi_2\xi_1 - 2\xi_1 - \xi_2 + 1 \\ 2\xi_1 + 2\xi_2 - 1 \end{bmatrix} \ ,$$
$$R(\xi_1, \xi_2) = \begin{bmatrix} 2\xi_1 + 2\xi_2 - 1 & -\xi_2\xi_1 + 2\xi_1 + \xi_2 - 1 \end{bmatrix} \ .$$

We generate from such a primal system vector-exponential data at the frequencies $(4, 3)$, $(5, 4)$, and $(9, \frac{1}{4})$, namely, $\left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix}\right] e^{4t_1} e^{3t_2}$, $\left[\begin{smallmatrix} 7 \\ 17 \end{smallmatrix}\right] e^{5t_1} e^{4t_2}$, and $\left[\begin{smallmatrix} 6 \\ -7 \end{smallmatrix}\right] e^{9t_1} e^{\frac{1}{4}t_2}$. Using the representation (3.4) of the dual system, we generate dual trajectories at the frequencies $(3, 1)$, $(\frac{1}{9}, 2)$, and $(7, \frac{1}{3})$, namely, $\left[\begin{smallmatrix} -9 \\ -11 \end{smallmatrix}\right] e^{3t_1} e^{t_2}$, $\left[\begin{smallmatrix} -47 \\ -31 \end{smallmatrix}\right] e^{\frac{1}{9}t_1} e^{2t_2}$, $\left[\begin{smallmatrix} -47 \\ -53 \end{smallmatrix}\right] e^{7t_1} e^{\frac{1}{3}t_2}$. It follows that $L^* J W = \left[\begin{smallmatrix} -161 & -250 & 23 \\ -497 & -856 & -65 \\ -783 & -1230 & 89 \end{smallmatrix}\right]$. It can be verified that the condition im $Y'^* \cap$ im $U'^* = \{0\}$ of Theorem 5.8 is satisfied by such data.

To compute a solution $\begin{bmatrix} \overline{S}_1 & \overline{S}_2 \end{bmatrix}$ to (5.12), we solve the Sylvester equations

$$M_i^* \overline{S}_i + \overline{S}_i \Lambda_i = \frac{1}{2} L^* J W \ , \tag{5.18}$$

$i = 1, 2$, obtaining

$$\overline{S}_1 = \begin{bmatrix} -\frac{23}{2} & -\frac{125}{8} & \frac{23}{24} \\ -\frac{4473}{74} & -\frac{1926}{23} & -\frac{585}{164} \\ -\frac{783}{22} & -\frac{205}{4} & \frac{89}{32} \end{bmatrix}, \; \overline{S}_2 = \begin{bmatrix} -\frac{161}{8} & -25 & \frac{46}{5} \\ -\frac{497}{10} & -\frac{214}{3} & -\frac{130}{9} \\ -\frac{2349}{20} & -\frac{1845}{13} & \frac{534}{7} \end{bmatrix}.$$

The following two matrices are solutions to (5.9):

$$G_1 = \begin{bmatrix} 4g_{11} & 5g_{12} & \frac{5g_{13}}{4} \\ 5g_{21} & 6g_{22} & \frac{9g_{23}}{4} \\ \frac{10g_{31}}{3} & \frac{13g_{32}}{3} & \frac{7g_{33}}{12} \end{bmatrix}, \; G_2 = \begin{bmatrix} -7g_{11} & -8g_{12} & -12g_{13} \\ -\frac{37g_{21}}{9} & -\frac{46g_{22}}{9} & -\frac{82g_{23}}{9} \\ -11g_{31} & -12g_{32} & -16g_{33} \end{bmatrix},$$

where $g_{ij} \in \mathbb{R}$ is a free parameter, $i, j = 1, \dots, 3$.

To check for the existence of parameters g_{ij} for which $S_i = \overline{S}_i + G_i$ satisfy the conditions of Theorem 5.8, we proceed as follows. Beginning with $(\mathtt{n}_1, \mathtt{n}_2) = (1, 0)$, we compute the Gröbner bases of the ideals generated by the minors of S_i of order $\mathtt{n}_i + 1$, $i = 1, 2$. It can be verified that for the minors of order 1 such bases consist only of the polynomial 1; consequently there exist no parameters g_{ij} for which S_1 and S_2 have ranks 1 and 0 or 0 and 1, respectively. This implies that no state models exist with $\mathtt{n}_1 = 1$ and $\mathtt{n}_2 = 0$ or $\mathtt{n}_1 = 0$ and $\mathtt{n}_2 = 1$. The Gröbner bases of the minors of order 2 do not consist of the unit polynomial only; moreover, the sum of the two ideals consisting of the order 2 minors also has a nontrivial Groebner basis; this implies that values of the parameters exist such that the matrices S_1 and S_2 have both rank 1.

Using Mathematica it can be computed that one set of values g_{ij} for which the minors of order 2 of S_1 and S_2 simultaneously annihilate, i.e., for which rank S_1 and rank S_2 both have rank 1 is

$$(g_{ij})_{i,j=1,\dots,3} = \begin{bmatrix} -3.02377 & -2.71645 & 6.25416 \\ -12.9249 & -11.8467 & 22.2606 \\ -11.5443 & -9.33238 & 42.4618 \end{bmatrix}.$$

The corresponding S_1 and S_2 are

$$S_1 = \overline{S}_1 + G_1 = \begin{bmatrix} -23.5951 & -29.2072 & 8.77604 \\ -125.071 & -154.819 & 46.5192 \\ -74.072 & -91.6903 & 27.5506 \end{bmatrix},$$
$$S_2 = \overline{S}_2 + G_2 = \begin{bmatrix} 1.04137 & -3.26841 & -65.8499 \\ 3.43585 & -10.7837 & -217.263 \\ 9.53762 & -29.9345 & -603.103 \end{bmatrix}.$$

For such matrices, conditions (1)–(2) of Theorem 5.8 hold. Computing rank-revealing factorizations $S_i = X_i'^* X_i$, $i = 1, 2$ via an SVD yields

$$X_1 = \begin{bmatrix} -9.49275 & -11.7506 & 3.53077 \end{bmatrix},$$
$$X_1' = \begin{bmatrix} 2.48559 & 13.1754 & 7.80301 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} -0.401179 & 1.25913 & 25.3682 \end{bmatrix},$$
$$X_2' = \begin{bmatrix} -2.59577 & -8.5644 & -23.774 \end{bmatrix}.$$

Via an SVD of $\begin{bmatrix} X_1'^\top & X_2'^\top & U'^\top \end{bmatrix}$ we compute a left inverse of $\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}$ satisfying the first equation in (5.13):

$$\begin{bmatrix} X_1'^* & X_2'^* \end{bmatrix}^\dagger = \begin{bmatrix} -0.344338 & 0.150758 & -0.0167127 \\ 0.555166 & -0.0558363 & -0.0825641 \end{bmatrix}.$$

The state and input matrices computed from the first two formulas (5.14) are

$$A' = \begin{bmatrix} 3.25981 & 1.55603 \\ 0.451739 & -0.169619 \end{bmatrix}, \ B' = \begin{bmatrix} -3.20108 \\ 1.50833 \end{bmatrix}.$$

Via an SVD of $\begin{bmatrix} U'^* & Y'^* \end{bmatrix}$ we compute F satisfying the second condition in (5.13):

$$F = \begin{bmatrix} -0.0115895 & 0.0450062 & -0.0427869 \end{bmatrix}.$$

The last two equations in (5.14) yield $C' = \begin{bmatrix} -1.30788 & -0.155653 \end{bmatrix}$ and $D' = 0.261076$. Such matrices correspond to the transfer function

$$G(\xi_1, \xi_2) = \frac{-1.69695 - 0.190492\xi_1 + 3.33557\xi_2 + 0.261076\xi_1\xi_2}{\xi_1\xi_2 + 0.169619\xi_1 - 3.25981\xi_2 - 1.25585},$$

which satisfies the interpolation conditions $G(4,3) = \frac{13}{2}$, $G(5,4) = \frac{17}{7}$, $G(9, \frac{1}{4}) = -\frac{7}{6}$ derived from the primal data directions $\left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 7 \\ 17 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 6 \\ -7 \end{smallmatrix}\right]$. ◻

We conclude this section discussing several issues. First, we examine alternative approaches to the procedure used in Example 2. We then show how dual trajectories can be constructed from primal ones. Subsequently, we discuss the relation of our approach to the 2D Loewner one and to the solution to the bivariate Nevanlinna interpolation problem of Agler and co-authors. Finally, we consider applying duality ideas to the identification of Fornasini–Marchesini i/s/o models.

Remark 4 (computational issues). In Example 2 we used a mixed symbolic-numerical approach to compute Roesser models. Anecdotal evidence obtained dealing with only a few more interpolation points suggests that such an approach is impractical for larger scale problems, since verifying the parametric rank conditions (1)–(2) in Theorem 5.8 using Gröbner bases is computationally rather intensive. The bottleneck is the calculation of minimal rank solutions S_1 and S_2 to (5.12); checking whether such solutions satisfy the additional conditions of Theorem 5.8 is a matter of standard computations.

One pair of solutions to (5.12) is straightforward to compute, see (5.18), and the set $\mathcal{G}$ in (5.17) is described by linear equations. Thus the computation of S_i can be reduced to an *affine rank minimization* problem:

$$\begin{aligned} &\text{Minimize rank } S_1 + \text{rank } S_2\,, \\ &\text{subject to } \mathcal{A}(S_1, S_2) = b\,, \end{aligned}$$

where $\mathcal{A}$ is a linear map, b is a vector obtained from L^*JR, and $\mathcal{A}(S_1, S_2) = b$ is a vector-formulation of (5.12). Several algorithms to solve this NP-hard problem are known; see, e.g., [26]. ◻

Remark 5 (data dualization via mirroring). In general it is difficult to obtain data from the dual system, and only data coming from the primal one are available (unless of course the two systems coincide—see [22, 24] for examples in the 1D case). We now describe the *mirroring* technique, already used in the 1D case (see [8, 9, 25]), to obtain dual data on the basis of primal ones. ◻

PROPOSITION 5.9. *Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ be controllable and $J \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be an involution. Let $\overline{w}e^{\lambda_1\cdot}e^{\lambda_2\cdot} \in \mathfrak{B}$, and let $\overline{v} \in \mathbb{C}^{\mathtt{w}}$ satisfy $\overline{v}^*\overline{w} = 0$. Then $J\overline{v}e^{-\lambda_1^*\cdot}e^{-\lambda_2^*\cdot} \in \mathfrak{B}^{\perp_J}$.*

Proof. Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi_1,\xi_2]$ and $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi_1,\xi_2]$ with $\mathtt{w}=\mathtt{p}+\mathtt{m}$ induce an image, respectively, kernel representation of $\mathfrak{B}$. Since $R(\xi_1,\xi_2)M(\xi_1,\xi_2) = 0_{\mathtt{p}\times\mathtt{m}}$, for all $(\lambda_1,\lambda_2) \in \mathbb{C}^2$, im $M(\lambda_1,\lambda_2) = (\text{im } R(\lambda_1,\lambda_2)^*)^\perp$, with orthogonality in the Euclidean sense in $\mathbb{C}^{\mathtt{w}}$. It follows that $J\overline{v}e^{-\lambda_1^*\cdot}e^{-\lambda_2^*\cdot} \in \text{im } JR(-\frac{\partial}{\partial t_1},-\frac{\partial}{\partial t_2})^\top = \mathfrak{B}^{\perp_J}$. ∎

Remark 6 (bivariate rational interpolation in the Loewner approach). In [2] a Loewner approach to bivariate rational interpolation is developed for SISO systems, based on the *Loewner matrix* of the data, which we now introduce through BDFs. Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi_1,\xi_2]$, $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi_1,\xi_2]$ induce an image, respectively, kernel representation of $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$. The multivariable representation of (5.2) is

$$R(-\zeta_1,-\zeta_2)M(\eta_1,\eta_2) = (\zeta_1+\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)+(\zeta_2+\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\,, \tag{5.19}$$

from which it follows that

$$\frac{R(-\zeta_1,-\zeta_2)M(\eta_1,\eta_2)}{(\zeta_1+\eta_1)(\zeta_2+\eta_2)} = \frac{\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)}{\zeta_2+\eta_2} + \frac{\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)}{\zeta_1+\eta_1}\,. \tag{5.20}$$

Since $w_i \in \text{im } M(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2})$ and $w_i' \in \text{im } R(-\frac{\partial}{\partial t_1},-\frac{\partial}{\partial t_2})^\top$ in (5.5), it follows that there exist latent variable trajectories $\ell_i(\cdot,\cdot) = \overline{\ell_i}e^{\lambda_1^i\cdot}e^{\lambda_2^i\cdot}$ and $\ell_i'(\cdot,\cdot) = \overline{\ell_i'}e^{\mu_1^i\cdot}e^{\mu_2^i\cdot}$ such that $\overline{w_i} = \left[\begin{smallmatrix}\overline{u_i}\\ \overline{y_i}\end{smallmatrix}\right] = M(\lambda_1^i,\lambda_2^i)\overline{\ell_i}$ and $\overline{w_i'} = \begin{bmatrix}\overline{u_i'}\\ \overline{y_i'}\end{bmatrix} = R(-\mu_1^i,-\mu_2^i)\overline{\ell_i'}$. Now substitute the values $(\lambda_1^i,\lambda_2^i)$ and (μ_1^j,μ_2^j) in place of (η_1,η_2) and (ζ_1,η_2), respectively, in (5.20) and multiply on the right by $\overline{\ell_i}$ and the left by $\overline{\ell_i'}$, respectively, obtaining $\mathbb{L} = \mathbb{P}_1+\mathbb{P}_2$, where the *Loewner matrix* $\mathbb{L}$ and $\mathbb{P}_i$, $i=1,2$, are $\mathbb{L}_{i,j} = \frac{\overline{w_i'}^\top\overline{w_j}}{(-\mu_1^j+\lambda_1^j)(-\mu_2^j+\lambda_2^j)}$, $(\mathbb{P}_1)_{n,m} = \frac{\overline{\ell_n'}^\top\Psi_1(-\mu_1^n,-\mu_2^n,\lambda_1^m,\lambda_2^m)\overline{\ell_m}}{-\mu_2^n+\lambda_2^m}$, $(\mathbb{P}_2)_{n,m} = \frac{\overline{\ell_n'}^\top\Psi_2(-\mu_1^n,-\mu_2^n,\lambda_1^m,\lambda_2^m)\overline{\ell_m}}{-\mu_1^n+\lambda_1^m}$, $i,j,n,m = 1,\ldots,N$. If all frequencies lie on the same side of the imaginary axis, e.g., the right-hand side one, such matrices are Gramians obtained integrating the corresponding BDFs on the external, respectively, latent variables, from $-\infty$ to 0, and (5.20) is the integral version of (5.2). In [2] bivariate Lagrange interpolation polynomial bases and kernels of appropriate submatrices of the Loewner matrix $\mathbb{L}$ are used to obtain generalized state-space models corresponding to *bidirectional* interpolating functions, i.e., satisfying the left and right interpolation conditions $G(\lambda_1^i,\lambda_2^i) = \frac{\overline{y_i}}{\overline{u_i}}$ and $G(\mu_1^i,\mu_2^i) = \frac{\overline{y_i'}}{\overline{u_i}'}$. ∎

Remark 7 (operator-theoretic approaches to bivariate interpolation). Agler, McCarthy, and others worked on discrete nD metric interpolation problems (see [1, 4, 6]) using operator-theoretic techniques; see also [5]. Interesting similarities exist between their formulas and ours; compare, e.g., [1, Thm. 11.49] with (5.19). A thorough investigation of the connections of such approaches with ours is a matter of pressing research, especially in view of the usefulness of BDF techniques in solving similar interpolation problems in the 1D case (see [8, 9, 25]). ∎

Remark 8 (Fornasini–Marchesini i/s/o models). The Roesser model is more advantageous than other classes of 2D quart-plane causal i/s/o representations to the application of our methodology; we examine the case of *second-order Fornasini-Marchesini models*

$$\begin{aligned}\frac{\partial^2}{\partial t_1\partial t_2}x &= A_1\frac{\partial}{\partial t_1}x + A_2\frac{\partial}{\partial t_2}x + A_3x + Bu\,,\\ y &= Cx+Du\,.\end{aligned} \tag{5.21}$$

It is a matter of straightforward verification to check that the dual of (5.21) is

$$\begin{aligned} \frac{\partial^2}{\partial t_1 \partial t_2} x' &= -A_1^\top \frac{\partial}{\partial t_1} x' - A_2^\top \frac{\partial}{\partial t_2} x' + A_3^\top x' + C^\top u' \,, \\ y' &= B^\top x' - D^\top u' \,. \end{aligned} \tag{5.22}$$

With tedious but straightforward manipulations it can be verified that if $\mathrm{col}(u, y, x)$ satisfies (5.21) and $\mathrm{col}(u', y', x')$ satisfies (5.22), then

$$y'^\top u + u'^\top y = \mathrm{div} \begin{bmatrix} x'^\top A_1 x + \frac{1}{2} x'^\top \left(\frac{\partial}{\partial t_2} x\right) - \frac{1}{2} \left(\frac{\partial}{\partial t_2} x'\right)^\top x \\ x'^\top A_2 x + \frac{1}{2} x'^\top \left(\frac{\partial}{\partial t_1} x\right) - \frac{1}{2} \left(\frac{\partial}{\partial t_1} x'\right)^\top x \end{bmatrix} . \tag{5.23}$$

Given Theorem 5.3 and the equivalence of Roesser and Fornasini–Marchesini models, it is not surprising that the external bilinear form is the divergence of a field involving the primal and the dual state. However, partial derivatives of the state are present, and thus the right-hand side of the matrix equation obtained from (5.23) for vector-exponential trajectories is more involved than the right-hand side of (5.12). $\square$

6. Conclusions. We considered two versions of the problem of modeling vector-exponential trajectories dependent on two independent variables with state-space models, and we provided two procedures to solve it, both essentially based on the factorization of constant matrices directly constructed from the data. Current research is aimed in several directions. First, we want to establish identifiability conditions based only on properties of the external data, since Proposition 5.5 falls short of being completely satisfactory. (See [15] on identifiability of nD systems.) Second, we need to develop a computationally efficient and numerically sound approach to the implementation of our procedure to compute Roesser models (see Remark 4). A third research direction is the identification problem from *general* (i.e., not polynomial vector-exponential) *discrete* data; cf. [22] for a BDF approach to such problem in the 1D case. On a longer horizon and a broader perspective, we want to investigate the application of our duality-based approach to *model reduction*. Finally, Roesser models only describe quarter-plane causal systems, and we need to generalize our results to more general notions of "causality" (e.g., those considered in [27] in the discrete case).

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